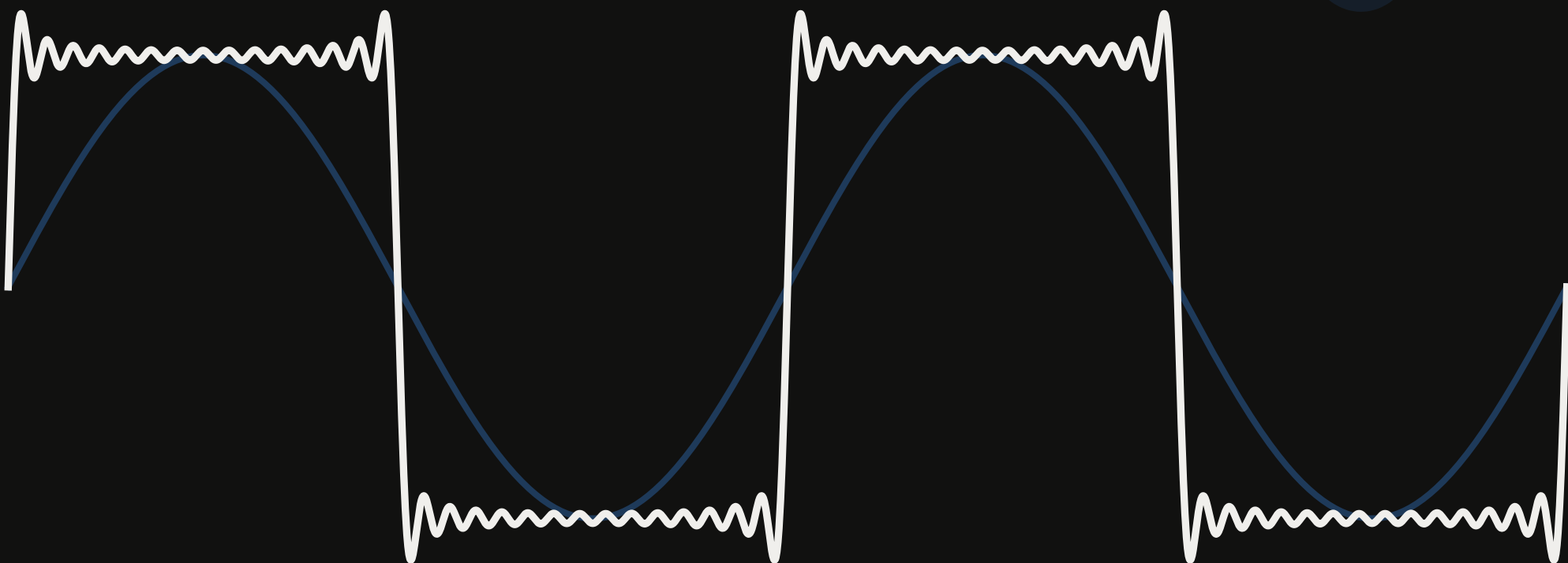


Why do we see harmonics?

Let's take some signals apart!

A mathematical answer,
with functions you can check yourself.



Fourier series · symmetry · smoothness · nonlinearity · the FFT

Swipe →

THE SETUP

Start with one assumption

A rotating machine running at constant speed produces motion that repeats every revolution. Mathematically, the signal is periodic:

$$f(t + T) = f(t)$$

T is the period (one revolution). It defines the fundamental frequency:

$$f_0 = \frac{1}{T}, \quad \omega_0 = \frac{2\pi}{T}$$

We assume nothing about the SHAPE of $f(t)$. It can be smooth, flattened, spiky, lopsided. Only that it repeats.

Everything that follows, including where harmonics appear and how strong they are, follows from that one line.

WHY INTEGER MULTIPLES

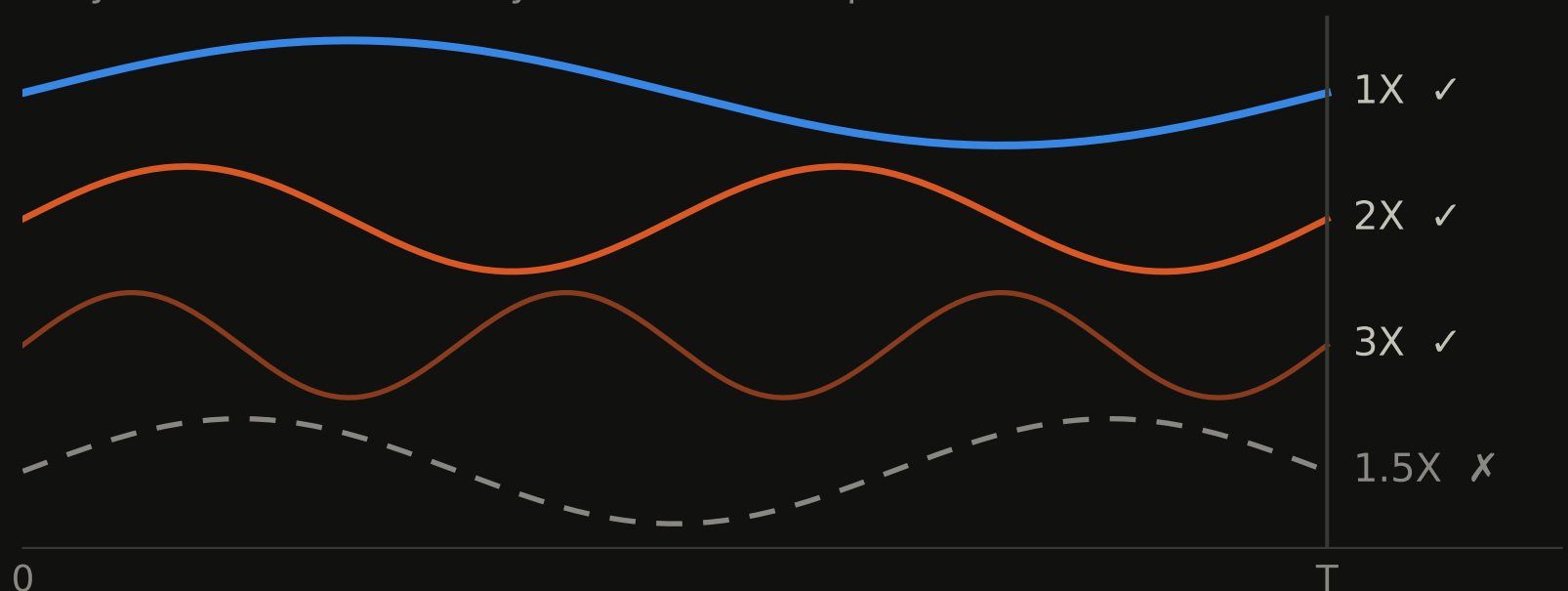
Which sines and cosines are allowed?

If a sine or cosine at frequency ω is part of a signal with period T , it must also repeat after T :

$$\sin(\omega(t + T)) = \sin(\omega t) \Rightarrow \omega T = 2\pi n$$

$$\Rightarrow \omega = n \omega_0, \quad n = 1, 2, 3, \dots$$

Only whole numbers of cycles fit into one period



Harmonics sit at EXACT integer multiples of running speed, not by convention, but because nothing else can fit.

FOURIER, 1822

The Fourier series

Any reasonable periodic function can be written as a sum of exactly those allowed sines and cosines:

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

Each coefficient measures how much of $f(t)$ looks like the n -th harmonic, a projection:

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega_0 t) dt, \quad b_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega_0 t) dt$$

This works because different harmonics are orthogonal: over one period they don't overlap at all.

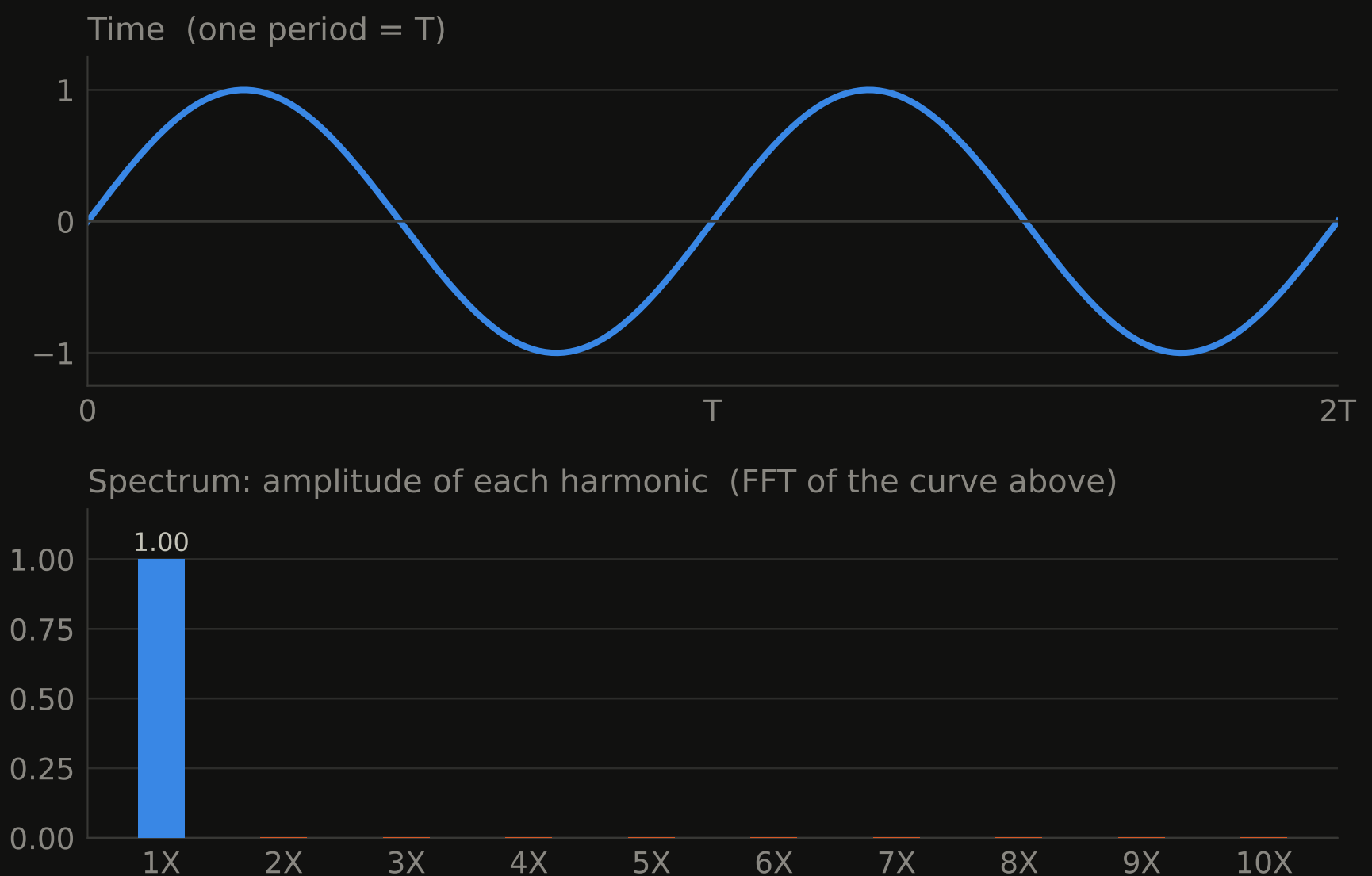
$$\int_0^T \sin(m\omega_0 t) \sin(n\omega_0 t) dt = 0 \quad (m \neq n)$$

The amplitude at nX on your spectrum is $\sqrt{a_n^2 + b_n^2}$. That's all a harmonic is.

BASELINE

Example 1: a pure sine

$$f(t) = \sin(\omega_0 t) \Rightarrow b_1 = 1, \text{ all others} = 0$$

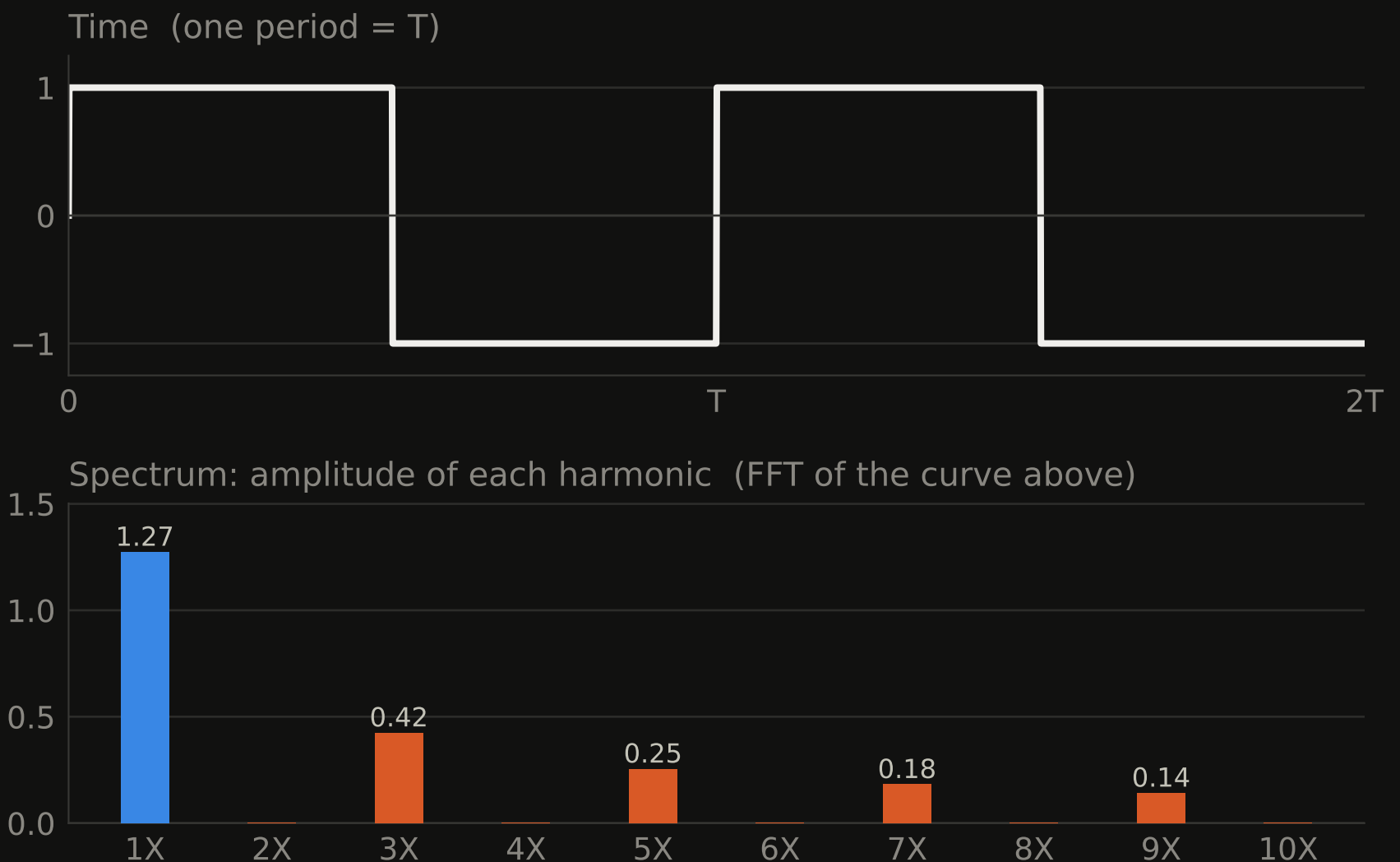


One line in the spectrum. Not approximately one: exactly one. A sine is already a harmonic, so the series has a single term.

ODD HARMONICS ONLY

Example 2: a square wave

$$\text{sgn}(\sin \omega_0 t) = \frac{4}{\pi} \left[\sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \dots \right]$$



Every odd harmonic, falling off as $1/n$. Check: $4/\pi = 1.27$, then $1.27/3 = 0.42$, $1.27/5 = 0.25$.
And not a single even one. Why?

HALF-WAVE SYMMETRY

Why only odd? Symmetry.

The square wave has half-wave symmetry: the second half-cycle is the first one flipped upside down.

$$f(t + T/2) = -f(t)$$

Now shift an even harmonic ($n = 2, 4, \dots$) by half a period:

$$\sin(n\omega_0(t + T/2)) = \sin(n\omega_0t + n\pi) = +\sin(n\omega_0t)$$

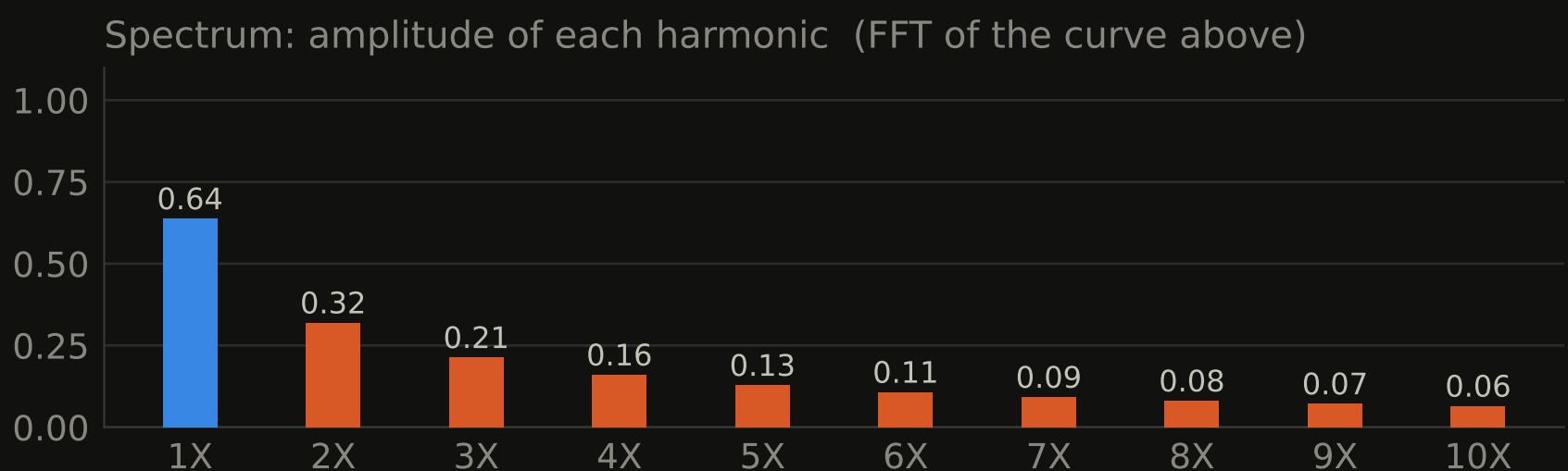
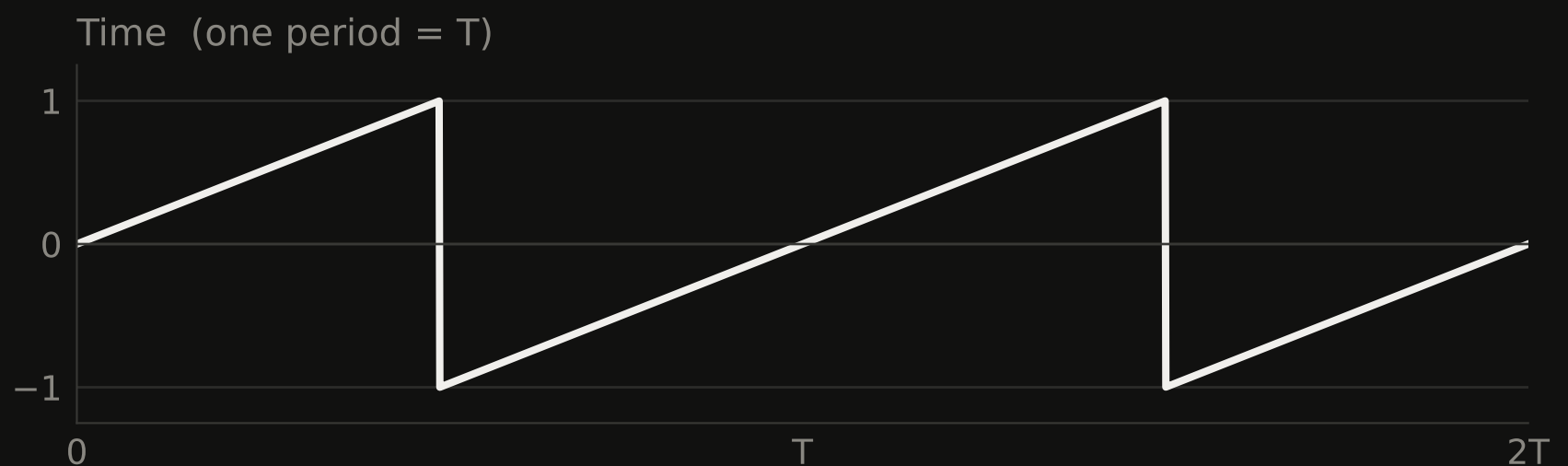
An even harmonic does NOT flip sign, so it can never contribute to a function that does. Its coefficient is forced to zero. Odd harmonics flip, so they survive.

**Symmetric distortion → odd harmonics only.
Break the symmetry → even harmonics appear.**

ALL HARMONICS

Example 3: a sawtooth

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\omega_0 t)}{n}$$

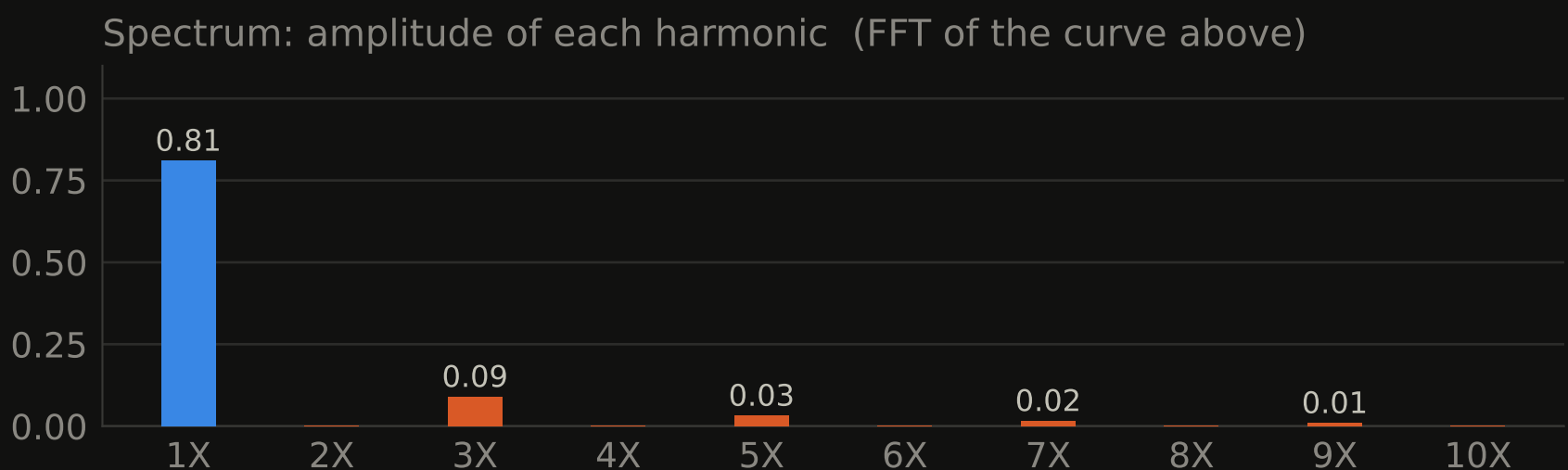
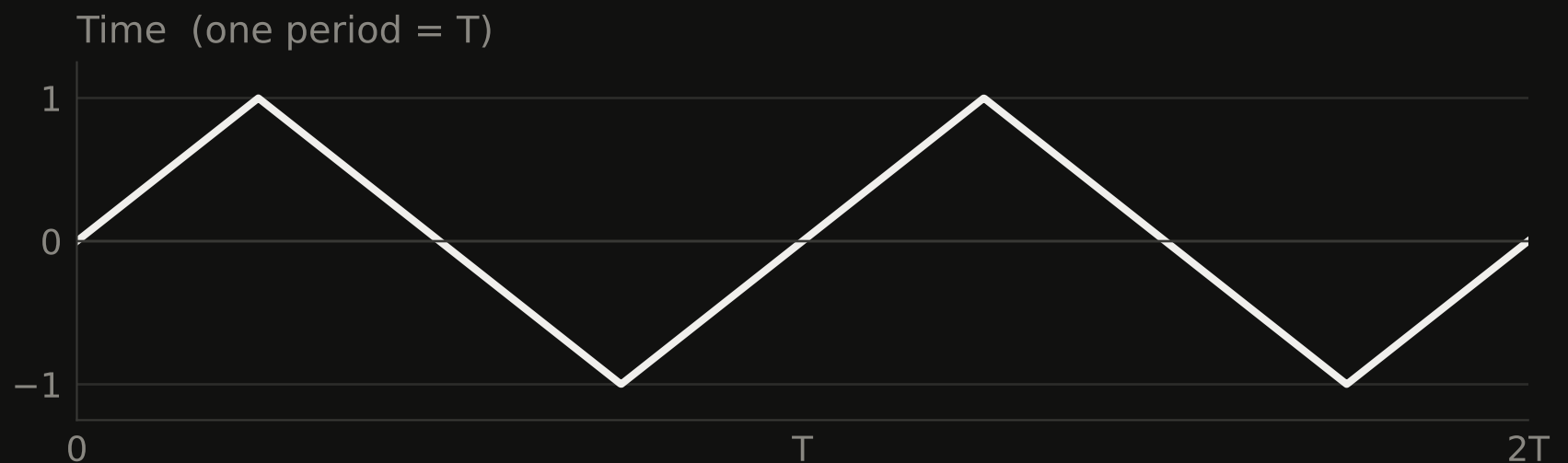


No half-wave symmetry, so every harmonic is present: 0.64, 0.32, 0.21, 0.16 ... still $1/n$, because the wave still has a jump.

SMOOTHER SHAPE

Example 4: a triangle wave

$$f(t) = \frac{8}{\pi^2} \sum_{n \text{ odd}} (-1)^{(n-1)/2} \frac{\sin(n\omega_0 t)}{n^2}$$



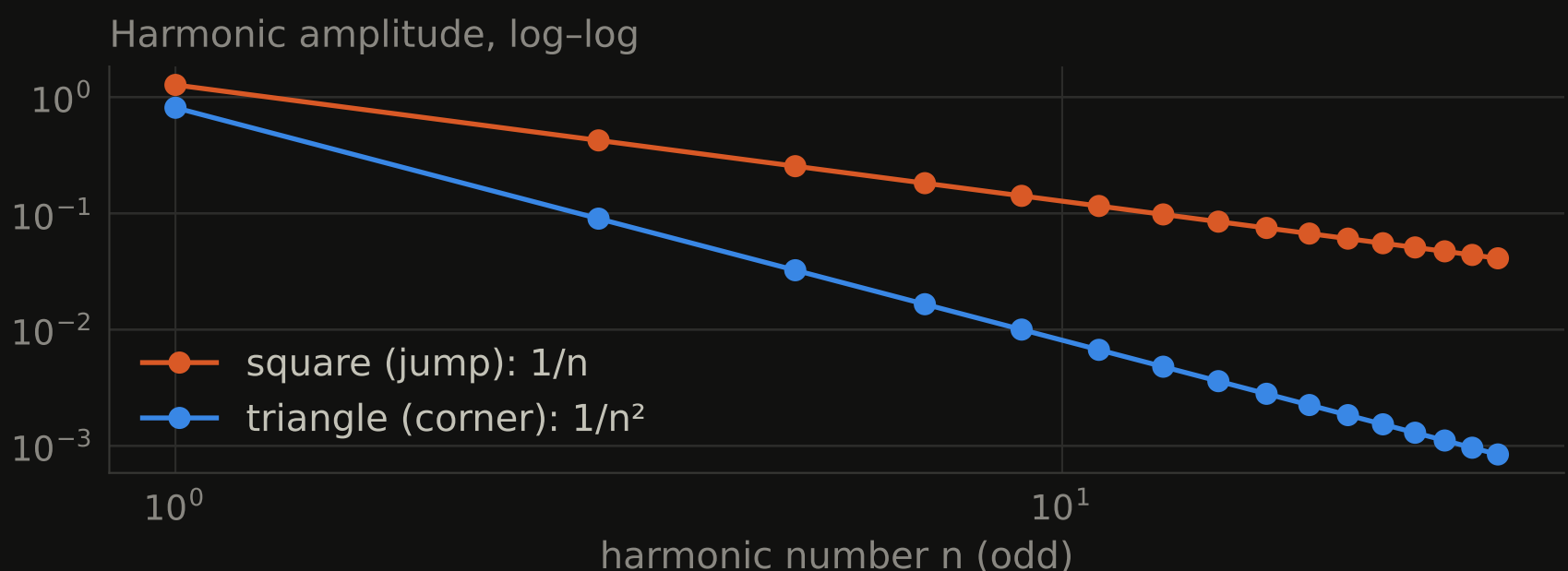
Same symmetry as the square wave, so again only odd harmonics. But no jump, only corners, and the harmonics fall as $1/n^2$: 3X is $1/9$ of 1X, 5X is $1/25$.

THE DECAY RULE

Sharper shape, more harmonics

Integrate the coefficient formula by parts: each step pulls out a factor $1/n$, until you hit a discontinuity. So:

Jump in $f(t)$	$\sim 1/n$	square, sawtooth
Corner (jump in f')	$\sim 1/n^2$	triangle, clipped sine
Jump in f''	$\sim 1/n^3$	smoother still
Infinitely smooth	$\ll 1/n^k$	a pure sine

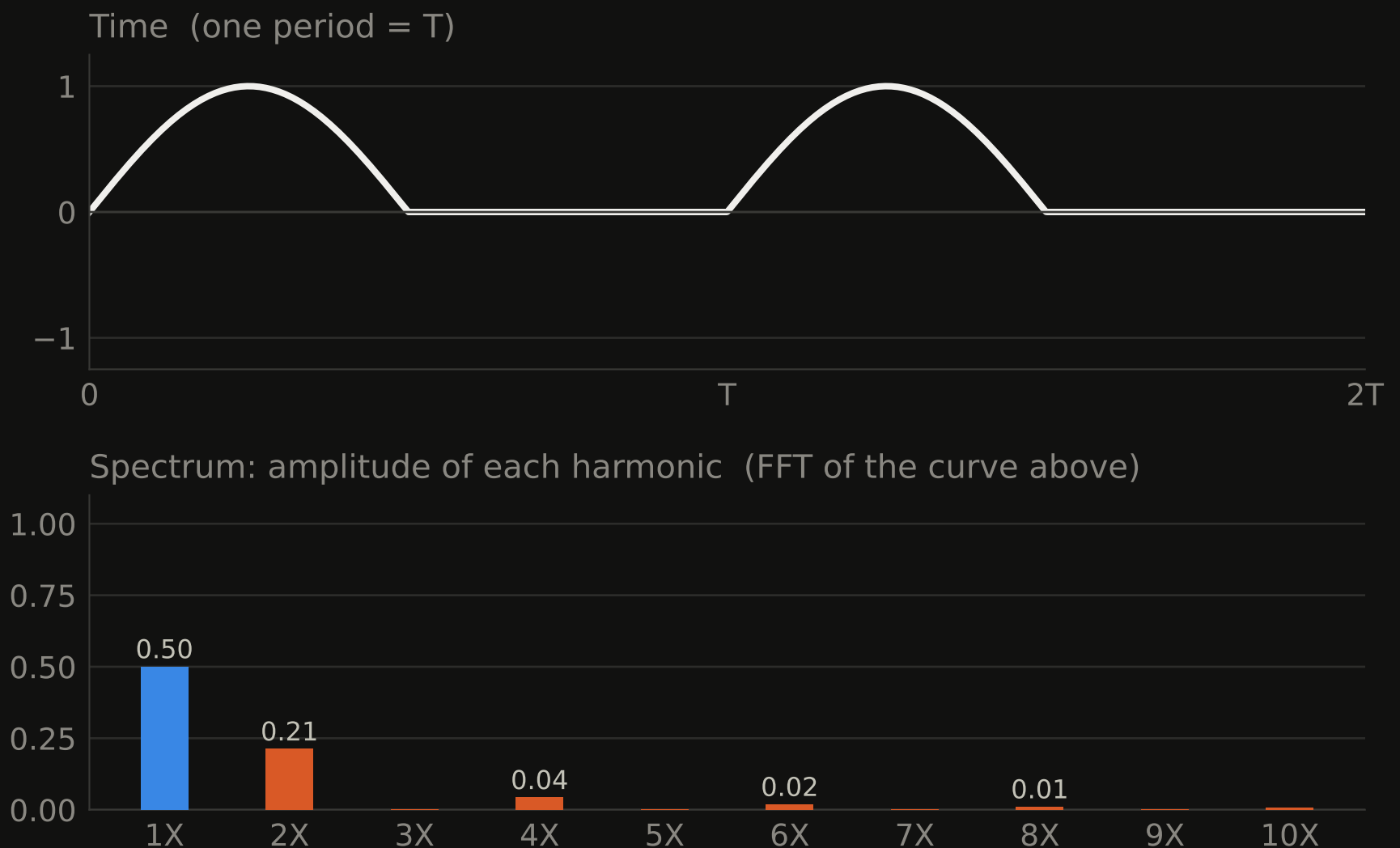


Sharp features in time live at high frequency. Impacts and sudden stops push energy far up the harmonic series.

ONE-SIDED TRUNCATION

Example 5: cut a sine on one side

$$\max(\sin \omega_0 t, 0) = \frac{1}{\pi} + \frac{1}{2} \sin \omega_0 t - \frac{2}{\pi} \sum_{n \text{ even}} \frac{\cos(n \omega_0 t)}{n^2 - 1}$$

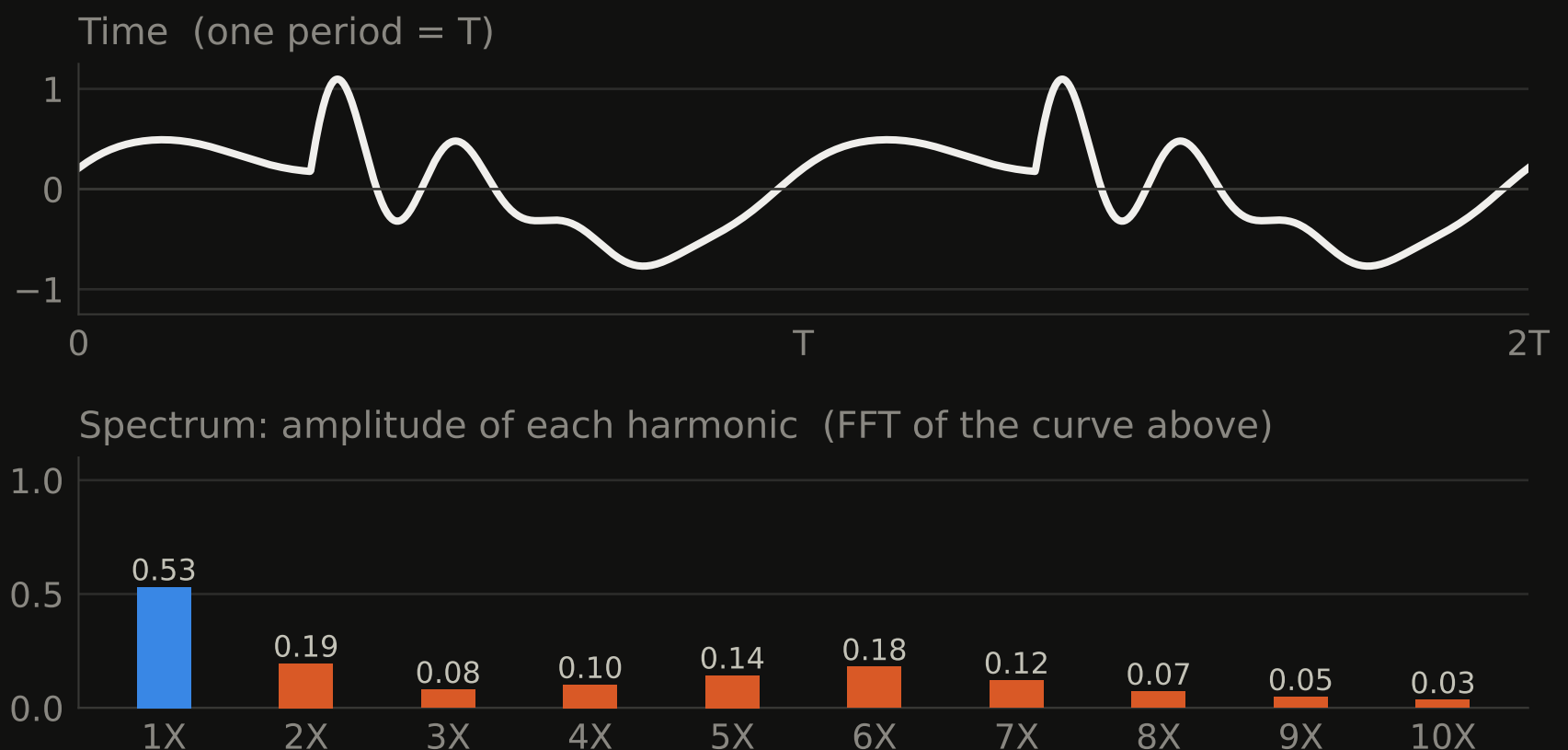


Flatten one side and the symmetry breaks: 2X, 4X, 6X appear immediately (0.21, 0.04, 0.02). This is the extreme case of motion that is limited in one direction only.

IN REAL LIFE

Example 6: a real machine signal

Real signals are messier: imbalance, misalignment, and an impact every revolution that rings and decays.



No neat formula, but it repeats every revolution, so it is still a Fourier series: imbalance on 1X, misalignment on 2X, the ringing impact spread across 4X–7X.

If it repeats and isn't a pure sine, you will see harmonics.

NONLINEARITY

Where harmonics are born

A linear system driven at ω_0 responds only at ω_0 . Harmonics need a nonlinearity: a response not proportional to the input.

$$y = x + \alpha x^2 + \beta x^3, \quad x = \sin \omega_0 t$$

Two identities you can check by hand:

$$\sin^2 \omega_0 t = \frac{1}{2} - \frac{1}{2} \cos 2\omega_0 t$$

$$\sin^3 \omega_0 t = \frac{3}{4} \sin \omega_0 t - \frac{1}{4} \sin 3\omega_0 t$$

The x^2 term creates $2X$ (and a DC shift). The x^3 term creates $3X$.

Even-order (asymmetric) → even harmonics.
Odd-order (symmetric) → odd harmonics.

Clearances, impacts, and stiffness that differs in each direction are all nonlinearities.

COOLEY & TUKEY, 1965

What the FFT actually computes

Your analyser doesn't see a formula, it sees N samples $x[m]$. The FFT is a fast way to compute the discrete Fourier transform:

$$X[k] = \sum_{m=0}^{N-1} x[m] e^{-i2\pi km/N}$$

If the record holds exactly P periods, harmonic n lands on bin $k = nP$, and $(2/N) \cdot |X[k]|$ equals the Fourier amplitude $\sqrt{a_n^2 + b_n^2}$. Same numbers as the series.

Two real-world caveats:

- Leakage: a non-integer number of periods smears peaks into neighbouring bins. A window (e.g. Hanning) controls it.
- Aliasing: harmonics above $f_s/2$ fold back to false frequencies. Anti-alias filtering prevents it.

TAKEAWAYS

What this means for the analyst

Harmonics aren't extra signals.

They are the Fourier description, in sines and cosines, of one periodic waveform that isn't a pure sine.

Position is forced by periodicity.

Exactly $n \times$ running speed. Peaks off those lines come from something else.

Which harmonics tells you symmetry.

Odd only: symmetric distortion. Evens present: something is one-sided.

How far they extend tells you sharpness.

Long harmonic families mean jumps or impacts in the waveform.

Rule out the measurement first.

Sensor saturation, a loose mount or a clipped input range distort the waveform too, and produce harmonics that aren't in the machine.

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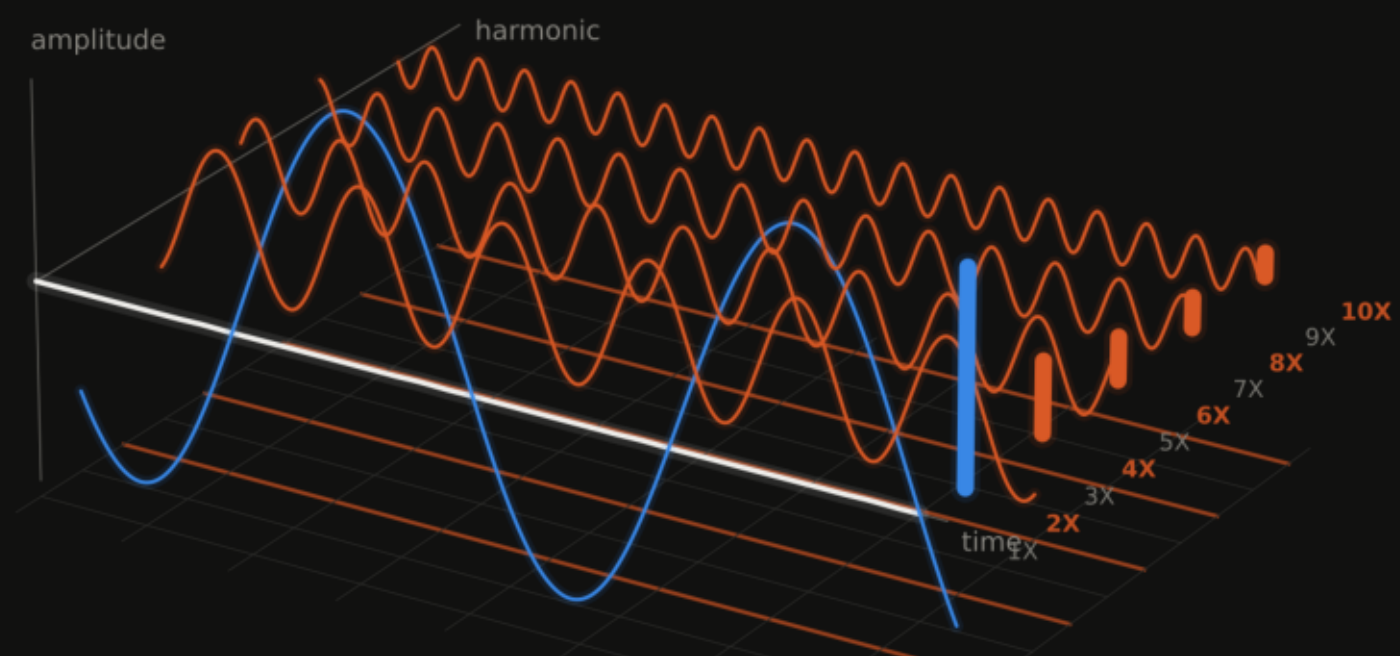
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COMING FRIDAY

Now watch them come apart

Every function in this carousel, pulled apart into its harmonic lanes and rebuilt, in 3D, with sound.



See you Friday!